Volume-1, Issue-3, Dec 2025

ISSN: 3107-4243 (Online)
World View Research Bulletin
An International Multidisciplinary Research Journal

Some Fixed Point Theorems in Fuzzy Metric Spaces

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Abstract

In many areas of mathematics, including analysis, topology, optimization, and the applied sciences, fixed point theory is essential. Fuzzy metric spaces have become a potent expansion of classical metric spaces in recent decades to deal with ambiguity and uncertainty. Establishing some novel fixed point theorems for contractive type mappings in fuzzy metric spaces is the goal of this research. Several established fixed point theorems in both classical and fuzzy contexts are generalized by the results presented. Additionally, illustrative examples are given to show how the results might be applied.

Keywords: *Fuzzy metric space, Fixed point, Contractive mapping, Continuous t-norm, Banach contraction principle.*

Introduction

One of the most important and often used fields of contemporary mathematical analysis is fixed point theory. A fundamental framework for resolving issues in nonlinear analysis, differential equations, optimization theory, control systems, economics and computer science is provided by the presence and uniqueness of fixed points for various classes of mappings. Since its introduction in metric spaces, the classical Banach contraction principle has sparked a great deal of study and many generalizations in many mathematical frameworks.

classical metric spaces often fail to adequately model real-world situations involving uncertainty, vagueness and imprecision. To overcome these limitations, fuzzy set theory, introduced by **Zadeh (1965)**, provided a robust mathematical apparatus to deal with uncertainty. Building upon this framework, **Kramosil and Michálek** introduced the notion of fuzzy metric spaces, which was later refined by George and Veeramani, making the concept more suitable for analytical and topological investigations.

Due to its usefulness in modeling complex systems with imperfect data, fuzzy metric spaces have garnered significant attention from researchers worldwide in recent decades, especially in India. The advancement of fixed point theory in fuzzy metric and related spaces has benefited greatly from the efforts of Indian mathematicians. By examining different contractive conditions and generalized mappings in fuzzy and probabilistic metric spaces, researchers including Bharucha-Reid, Cho, Pant, Mishra, Singh, and Jain have extended classical conclusions to more adaptable and practical frameworks.

Furthermore the importance of entire fuzzy metric spaces in the existence and uniqueness of fixed points has been highlighted in recent Indian research, with a special emphasis on mappings that meet nonlinear contractive requirements controlled by control functions. These studies are driven by applications in computational intelligence, fuzzy differential equations and decision-making models where uncertainty is a major factor.

Even though there has been a lot of development, there is still a lot of room to create new fixed point theorems in fuzzy metric spaces under more generalized contractive conditions and weaker assumptions. The current study seeks to develop some novel fixed point theorems for self-mappings in fuzzy metric spaces, motivated by these facts and the expanding corpus of Indian and worldwide research. The discovered results expand various existing conclusions in the fuzzy metric setting and generalize classical fixed point theorems.

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Preliminaries

In this section, we recall some basic definitions and results that will be used throughout the paper.

Definition 1 (Continuous t-norm)

A binary operation

$$*: [0,1] \times [0,1] \rightarrow [0,1]$$

is called a **continuous t-norm** if it satisfies the following conditions:

1. is associative and commutative
2. is continuous
3. $a * 1 = a \; for \; all \; a \in [0,1]$
4. $a * b \leq c * d \; whenever \; a \leq c and \; b \leq d$

Examples:

$$a * b = ab (product \; t - norm)$$

$$a * b = min \{a, b\}$$

Definition 2 (Fuzzy Metric Space)

Let X be a non-empty set and * be a continuous t-norm. A function

$$M: X \times X \times (0, \infty) \rightarrow [0,1]$$

is called a **fuzzy metric** on X if for all x,y,z ∈ Xx,y,z and t,s > 0:

1. M(x,y,t) > 0
2. M(x,y,t) = 1 if and only if x=y
3. M(x,y,t) = M (y,x,t)
4. M(x,z,t+s) ≥ M(x,y,t)∗M(y,z,s)

5. M(x,y,t) M (x,y,t) M(x,y,t) is continuous in t

Then the triple (X,M,∗) is called a **fuzzy metric space**.

Definition 2 (Convergence and Cauchy Sequence)

A sequence $\{x_n\}$ in a fuzzy metric space (X,M,∗) is said to **converge** to x ∈ X if:

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \qquad \forall t > 0$$

The sequence is called **Cauchy** if:

$$\lim_{n\to\infty} M(x_n, x_m, t) = 1 \qquad \forall t > 0$$

A fuzzy metric space is **complete** if every Cauchy sequence converges.

Fixed Point Theorems

Definition 3.1 (Fuzzy Contractive Mapping)

A mapping T: $X \to X$ is said to be a fuzzy contraction if there exists a constant $k \in (0, 1)$ such that:

$$M\left(T_x, T_y, k_t\right) \geq M(x, y, t) \qquad \forall_{x,y,t} \in X, t > 0$$

Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ by:

$$x_{n+1} = Tx_n \qquad for n \geq 0$$

Using the contractive condition, we have:

$$M(x_{n+1,xn}, kt) = M(Tx_n, Tx_{n-1}, kt) \geq M(x_n, x_{n-1}, t)$$

By iteration, it follows that:

$$M(x_n, x_{n+1}, k^n t) \geq M(x_0, x_1, t)$$

As $n \to \infty$, $k^n\ t \to 0$, hence:

$$\lim_{n\to\infty} M(x_n, x_{n-1}, t) = 1$$

Thus, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$

such that:

$$\lim_{n\to\infty} x_n = x.$$

By continuity of T, we get:

$$T_x = x$$

Uniqueness comes naturally. Assume,

$$\mathrm{M(x, y, kt)} = M(T_x, T_y, kt \geq M(x.y, t)$$

which means M(x, y, t) = hence $x = y$.

4. Generalized Fixed Point Theorem

Theorem 4.1

Let $(X, M, *)$ be a complete fuzzy metric space. Suppose T: $X \to X$ satisfies:

$$M(Tx,Ty,t) \geq \phi(M(x, y, t))$$

where ϕ:[0, 1] → [0, 1]

ϕ:[0,1]→[0,1] is a continuous increasing function such that:

$$(s) > s \text{ for all } s \in (0,1)$$

Then T has a unique fixed point.

Following similar steps as in Theorem and using the properties of ϕ, the sequence $\{x_n\}$ defined by $x_{n=1} = Tx_n$ converges to a unique fixed point.

Example

Let X=ℝ, define:

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

using the t-norm $a * b = ab$

Define $T: X \to X$ as follows:

$$T(x) = \frac{x}{2}$$

Then:

$$M(T_x, T_y, t) = \frac{t}{t + \frac{|x-y|}{2}} \geq M(x, y, 2t)$$

As a result, every requirement of Theorem Above is met, and T has a single fixed point at $x = 0$.

Conclusion

The existence and uniqueness of fixed points for a class of contractive type mappings in complete fuzzy metric spaces have been examined in this study. A number of fixed point theorems have been established by using the framework of continuous t-norms and generalized contractive conditions. The findings of this study broaden and extend several fixed point conclusions in fuzzy metric spaces as well as the traditional Banach contraction principle.

This work creates a number of opportunities for more study. Multivalued mappings, intuitionistic fuzzy metric spaces, probabilistic metric spaces, and other generalized fuzzy structures can also benefit from the current findings.

The application of these fixed point theorems to practical issues in engineering, economics, and artificial intelligence may also be the subject of future research. As a result, the results of this work offer a basis for further theoretical and practical investigation and significantly add to the expanding body of literature on fixed point theory in fuzzy metric spaces.

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